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tubular necrosis and interstitial edema. Again, atoxyl causes extensive hemorrhage extending from the boundary zone through the medulla and only slightly into the cortex, the outer rim of which remains pale; there is marked degeneration and necrosis of tubular epithelium. These atoxyl kidneys which are outwardly pale combine in a peculiar form the essential features of both the red and the pale kidneys. In like manner, arsenophenylglycine produces kidney lesions of a combined type and while tubular degeneration and necrosis are dominant there is usually some congestion and hemorrhage in the boundary zone and medulla—more rarely in the cortex.

The other extreme in the action of arsenicals upon the kidney is exemplified by arsacetin which produces a typically pale kidney. While congestion and hemorrhage are still apparent to a minor degree in the boundary zone of these kidneys the vascular injury is so completely overshadowed by the injury to the tubular epithelium as to leave no doubt as to the dominance of tubular injury. Further, the prompt and vigorous regeneration of the tubular epithelium indicates that the extensive necrosis produced by arsacetin can not be regarded as a secondary anemic phenomenon.

It is certain, therefore, that all arsenicals do not produce renal lesions that are identical either in character or distribution but that this group of substances includes agents producing a so-called tubular nephritis as well as those producing a vascular nephritis, and that these wide differences in the pathogenic action of different compounds of arsenic are explainable only upon the basis of their chemical constitution.

SEVEN POINTS ON A TWISTED CUBIC CURVE

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Six points in space, barring special situations, determine a twisted cubic curve. From any seventh point of the curve those six are projected by six generators of a quadric cone. For any seven points of a cubic curve there is accordingly a symmetric set of seven cones; and it is well known that seven points giving rise to two such cones are on a cubic curve, and so give rise to five more cones. This is the only current theorem on seven points of a twisted cubic. Concerning eight points there is the elegant theorem of von Staudt, that if two tetraedrons have eight points of a twisted cubic for vertices, their eight faces osculate a second cubic curve. I propose to demonstrate a theorem whose formulation resembles the latter, while like the former it relates to the

fewest possible points, seven. It is this: *If seven points on a twisted cubic be joined, two and two, by twenty-one lines, then any seven planes that contain these twenty-one lines will osculate a second cubic curve.*

The proof to be given is analytic, consisting in a repeated application of Pascal's theorem concerning a hexagon inscribed in a conic, adapted to the six projecting lines from a seventh point by means of Clebsch's so called *Ränderungs-princip*.

Pascal's theorem may be expressed in notation readily understood, for six points a, b, c, d, e, f of a conic (coordinates (a_1, a_2, a_3) etc.) by the vanishing of the determinant

$$\begin{vmatrix} (ab), (de)_1 & (bc), (ef)_1 & (cd), (fa)_1 \\ (ab), (de)_2 & \dots\dots\dots & \dots\dots\dots \\ (ab), (de)_3 & \dots\dots\dots & \dots\dots\dots \end{vmatrix}$$

or any one of the 59 equivalent forms. This is reducible to the equation, in three-rowed determinants of point-coordinates,

$$(abe) (cde) (adf) (bcf) = (abf) (cdf) (ade) (bce). \quad (1)$$

(See *Encyklopädie der mathematischen Wissenschaften*, 3 $\frac{2}{2}$, §7.)

Clebsch's *Ränderungs-princip* applied to this equation gives the condition that six points in space (a_1, a_2, a_3, a_4) , etc., or briefly a, b, c, d, e, f , shall be projected from a seventh point (g_1, g_2, g_3, g_4) by six generators of a quadric cone, viz.,

$$(abeg) (cdeg) (adfg) (bcfg) = (abfg) (cdfg) (adeg) (bceg). \quad (2)$$

If the seven points are on a twisted cubic, this relation may be written in $7 \cdot 15 \cdot 3$ different ways, according to the choice of the seventh point, g , the two points e and f , and the two ways of forming pairs from the remaining four points a, b, c, d . Let equation (2) denote all of this type.

Seven planes, 1, 2, 3, 4, 5, 6, 7 may be determined in 30 different ways so as to contain the 21 lines that join pairs of the points a, b, c, d, e, f, g . Any one such choice constitutes a *triad system* on those seven letters, since its seven sets of three points in a plane must contain every pair of points once and only once. Fix one set thus, planes, 1, 2, 3, 4, 5, 6, 7 containing the points $ade, afg, bdf, beg, cdg, cef, abc$, respectively. These seven planes, in order to osculate a twisted cubic, must satisfy conditions on planes precisely dual to the 315 conditions of type (2) on points. On account of their similarity we need to verify only one. This is a slightly tedious reckoning, which we give *in extenso*.

The question shall be: Does the plane 7 intersect the six planes 1, 2, 3, 4, 5, 6 in six tangents to a conic? Is the following condition verified?

$$(1257) (3457) (1467) (2367) = (1267) (3467) (1457) (2357). \quad (3)$$

Now it is necessary to express the coordinates of each plane as three-rowed determinants in the coordinates of the three points which it contains, to simplify, and compare the left hand product of four determinants in (3) with that on the right, and to show that equations (2) above will suffice to prove the equality so reduced. On the left:

$$\begin{aligned}
 (1257) &\equiv (ade)_1 (afg)_2 (cdg)_3 (abc)_4 \\
 &\equiv (adeb) (afgc) (cdga) - (adec) (afgb) (cdga) \\
 &\equiv (cdga) \{ (adeb) (afgc) - (adec) (afgb) \}, \\
 (3457) &\equiv (cdgb) \{ (bdfc) (bega) - (bdfa) (becg) \}, \\
 (1467) &\equiv (adeb) (becg) (cefa) + (adec) (bega) (cefb), \\
 (2367) &\equiv (afgb) (bdfc) (cefa) + (afgc) (bdfa) (cefb).
 \end{aligned}$$

On the right:

$$\begin{aligned}
 (1267) &\equiv (cefa) \{ (adeb) (afgc) - (adec) (afgb) \}, \\
 (3467) &\equiv (cefb) \{ (bdfc) (bega) - (bdfa) (becg) \}, \\
 (2357) &\equiv (afgb) (bdfc) (cdga) + (afgc) (bdfa) (cdgb), \\
 (1457) &\equiv (adeb) (becg) (cdga) + (adec) (bega) (cdgb).
 \end{aligned}$$

Notice that on the left two factors in the $\{ \}$ are identical with two in $\{ \}$ on the right. Exclude these, when multiplication gives in each member four terms. Two terms on the left are identical with terms on the right. Remove these, and transpose so as to exhibit the factor

$$(cefa) (cdgb) - (cefb) (cdga)$$

on each side. Neglect this, and our question is reduced to the following. Is it true that

$$\begin{aligned}
 (abfg) (abde) \cdot (bcdf) (bceg) \cdot (acdg) (acef) \\
 = (abdf) (abeg) \cdot (bcdg) (bcef) \cdot (acde) (acfg) ? \quad (4)
 \end{aligned}$$

Two applications of relations like (2) prove the truth of this, for example we use first b , then c , as seventh point like the g of (2);

$$(abfg) (abde) (bcdf) (bceg) = (abdf) (abeg) (bcfg) (bcde),$$

and

$$(bcfg) (bcde) (acdg) (acef) = (bcdg) (bcef) (acfg) (acde).$$

These two Pascalian relations upon the seven points are part of the hypothesis, hence the Pascalian equation (3) upon the seven planes is verified and the theorem is proved.

It is interesting to restate this theorem somewhat more fully. The converse is true by duality, from seven planes to any seven points that lie on all 21 of their lines of intersection. Consequently

If seven planes intersect three and three in seven points of a twisted cubic, each of the 29 other sets of seven points that contain all the 21 lines of intersection of those planes is upon another twisted cubic curve.